Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B (Proc. Suppl.) 186 (2009) 117–120
NUCLEAR PHYSICS B PROCEEDINGS SUPPLEMENTS
www.elsevierphysics.com

Exact Mesonic Eightfold Way From Dynamics and Confinement in Strongly Coupled Lattice QCD

A. Francisco Neto[a], M. O'Carroll[b] and P.A. Faria da Veiga[b] *

[a]Campus Alto Paraopeba, UFSJ
C.P. 131, 36.420-000 Ouro Branco MG, Brazil

[b]Departamento de Matemática Aplicada e Estatística, ICMC-USP
C.P. 668, 13.560-970 São Carlos SP, Brazil

We review our results on the exact determination of the mesonic eightfold way from first principles, directly from the quark-gluon dynamics. For this, we consider an imaginary-time functional integral formulation of $3+1$ dimensional lattice QCD with Wilson action, three flavors, $\mathrm{SU}(3)_f$ flavor symmetry and $\mathrm{SU}(3)_c$ local gauge symmetry. We work in the strong coupling regime: a small hopping parameter $\kappa > 0$ and a much smaller plaquette coupling $\beta > 0$. By establishing a Feynman-Kac formula and a spectral representation to the two-meson correlation, we provide a rigorous connection between this correlation and the one-meson energy-momentum spectrum. The particle states can be labeled by the usual $\mathrm{SU}(3)_f$ quantum numbers of total isospin I and its third-component I_3, the quadratic Casimir C_2 and, by a partial restoration of the continuous rotational symmetry on the lattice, as well as by the total spin J and its $z-$component J_z. We show that, up to near the two-meson energy threshold of $\approx -4\ln\kappa$, the spectrum in the meson sector is given only by isolated dispersion curves of the eightfold way mesons. The mesons have all asymptotic mass of $-2\ln\kappa$ and, by deriving convergent expansions for the masses both in κ and β, we also show a κ^4 mass splitting between the $J = 0, 1$ states. The splitting persists for $\beta \neq 0$. Our approach employs the decoupling of hyperplane method to uncover the basic excitations, complex analysis to determine the dispersion curves and a correlation subtraction method to show the curves are isolated. Using the latter and recalling our similar results for baryons, we also show confinement up to near the two-meson threshold.

1. Introduction

We rigorously determine the one-particle spectrum exactly for a three-flavor lattice QCD model with Wilson action [1,5] defined by using an imaginary-time functional integral formulation in $3+1$ dimensions, at the strong coupling regime, i.e. with the a plaquette coupling $\beta \equiv 1/g_0^2$ and a hopping parameter κ verifying $0 < \beta \ll \kappa \ll 1$. This problem is a part of an ongoing program aiming at obtaining the low-lying spectrum, i.e. the existence of hadrons and two-hadron bound states from the QCD dynamics. Lattice QCD with strong coupling can be analytically tractable by rigorous methods and furnishes a good qualitative description of the expected particle content of the continuum model. An underlying suitable quantum mechanical Hilbert space $\mathcal{H}$ can be constructed, energy-momentum (E-M) operators can be defined and the thermodynamical limit can be controlled [1]. More recently, we showed that the the low-lying E-M spectrum consists solely of the eightfold way particles [2], i.e. the eightfold way mesons [3] and baryons [4], and no glueballs. If $0 < \kappa \ll \beta$ then the low-lying spectrum consists only of glueballs. We obtained the leading order particle dispersion curves and showed the particle masses are jointly analytic in κ and β In this short note, we review some of the main points of our method. We concentrate on the mesonic eightfold way and refer to Ref. [3] for details.

Of course, this is not the first time the existence of composed hadrons is treated in the literature. Many papers considered this problem, especially in the 80's [6], using unjustified spectral methods, using e.g. *only* decay properties of correla-

*Partially supported by CNPq, FAPEMIG, PROPE-UFSJ. E-mail: afneto@fisica.ufmg.br, veiga@icmc.usp.br.

0920-5632/$ – see front matter © 2008 Published by Elsevier B.V.
doi:10.1016/j.nuclphysbps.2008.12.027

tions. Unfortunately, for example, these methods do not distinguish between spectral points associated to particles or points in a band, nor can they account for particle multiplicities with eventual small splittings between them. In our case, it is important to emphasize that our results are based on spectral representations for the two-point correlations, that particles are detected through *isolated* E-M dispersion curves and that a subtraction method allows us to show that the dispersion curves are isolated up to near the two-particle energy threshold. Hence, combining our present eightfold way meson results with the ones for baryons [4], we show confinement up to near the two-meson threshold. Besides these new and mathematically firm results, giving more support to the lattice approach and the Gell-Mann and Ne'eman quark model, and which can incorporate more quark flavors to support e.g. the existence of unobserved particles as the scalar mesons and heavy baryons, we observe that our method also fits into the analysis of the two-hadron spectrum using the one-particle results and a lattice version of the Bethe-Salpeter equation. In this way, our program will shortly consider important questions such as the theoretical existence of exotic states, such as tetraquarks and pentaquarks.

2. The model and physical Hilbert space $\mathcal{H}$

We use the same model presented in Refs. [3,4]. The model is the $\mathrm{SU}(3)_f$ lattice QCD model with the partition function given formally by $Z = \int e^{-\mathcal{S}(\psi,\bar\psi,g)} d\psi d\bar\psi d\mu(g)$, and for a function $F(\psi,\bar\psi,g)$, the normalized correlations are

$$\langle F\rangle = \frac{1}{Z}\int F(\psi,\bar\psi,g)e^{-\mathcal{S}(\psi,\bar\psi,g)} d\psi d\bar\psi d\mu(g).$$

The model action $\mathcal{S}(\psi,\bar\psi,g)$ is Wilson's action [5] given by (with $v = u+\epsilon e^\rho$, e^ρ being a lattice unit vector)

$$\mathcal{S} = \frac{\kappa}{2}\sum \bar\psi_{a,\alpha,f}(u)\,\Gamma^{\epsilon e^\rho}_{\alpha\beta} U(g_{u,v})_{ab}\,\psi_{b,\beta,f}(v) + \sum_{u\in\mathbf{Z}^4_o} \bar\psi_{a,\alpha,f}(u) M_{\alpha\beta}\psi_{a,\beta,f}(u) - \frac{1}{g_0^2}\sum_p \chi(g_p),$$

where the first sum is over $u \in \mathbf{Z}^4_o$, $\epsilon = \pm 1$, $\rho = 0,1,2,3$ and over repeated indices. Calling 0 the temporal direction, the lattice is given by $\mathbf{Z}^4_o$, where $u = (u^0,\vec u) = (u^0,u^1,u^2,u^3) \in \mathbf{Z}^4_o \equiv \mathbf{Z}_{1/2}\times\mathbf{Z}^3$, where $\mathbf{Z}_{1/2} = \{\pm 1/2, \pm 3/2, \ldots\}$. For each site $u \in \mathbf{Z}^4_o$, there are fermionic fields, represented by Grassmann variables, $\psi_{a,\alpha,f}(u)$ and $\bar\psi_{a,\alpha,f}(u)$ which carry a Dirac spin $\alpha = 1,2,3,4$, a color $a = 1,2,3$ and flavor $f = u,d,s = 1,2,3$ index. $\Gamma^{\pm e^\rho} = -1 \pm \gamma^\rho$, where the γ^ρ are the Dirac 4×4 matrices. For each oriented bond on the lattice we associate a gauge group element $U(g_{u+e^\rho,u}) \equiv U(g_{u,u+e^\rho})^{-1}$, p in the action $\mathcal{S}$ is a plaquette and $\chi(g_p)$ is the real part of the trace of the oriented product of U's in a plaquette. The measure $d\mu(g)$ is the product measure over non-oriented bonds of normalized $\mathrm{SU}(3)_c$ Haar measures and the integrals over Grassmann fields are Berezin integrals (see [1]).

Associated with our model, there is an underlying quantum mechanical physical Hilbert space $\mathcal{H}$ (see Ref. [1]). In this way, we can establish a connection between expectations and inner products in $\mathcal{H}$ by the following Feynman-Kac (F-K) formula. For F and G only depending on coordinates with $u^0 = 1/2$, we have

$$(G, \check T_0^{x^0}\check T_1^{x^1}\check T_2^{x^2}\check T_3^{x^3} F)_{\mathcal{H}} = \langle [T_0^{x^0}\vec T^{\vec x} F]\Theta G\rangle \qquad (1)$$

where $\vec T^{\vec x} = T_1^{x^1}T_2^{x^2}T_3^{x^3}$ is translation in space by $\vec x$, $\check{}$ is used to denote Hilbert space operators and Θ is an anti-linear operator which involves time reflection. As in Refs. [3,4], by taking the Fourier transform of the F-K formula, a connection is rigorously established between the spectrum of the self-adjoint E-M operators $H \geq 0$ and $P^{i=1,2,3}$, and singularities of the two-meson correlation. Here, we have $\check T_0^2 = e^{-2H}$ and $\check T_j = e^{iP^j}$.

We are interested in the joint spectrum of H and $\vec P = (P^1,P^2,P^3)$. Below, we show how our methods determine the one-meson spectrum and also describe briefly our main results.

3. Detection of the eightfold way mesons

To detect mesons, we introduce a general two-point correlation, with arbitrary M, L in the subspace $\mathcal{H}_e \subset \mathcal{H}$ generated by an even number of Grassmann variables. Using the order $u^0 < v^0$

value to extend the values to $u^0 = v^0$, we define

$$\mathcal{G}_{ML}(u,v) \equiv \begin{cases} \langle \Theta M(u)L(v)\rangle , & u^0 \le v^0 \\ \langle M(u)\Theta L(v)\rangle^* , & u^0 > v^0 . \end{cases} \quad (2)$$

By Eq. (1), the correlation in Eq. (2) admits the spectral representation, for $x^0 \neq 0$, $x = (x^0, \vec{x}) = v - u$, $\mathcal{G}_{ML}(x) = \mathcal{G}_{ML}(0, v-u) = \mathcal{G}_{ML}(u,v)$,

$$\mathcal{G}_{ML}(x) = \int_{-1}^{1}\int_{T^3} \lambda_0^{|x^0|-1} e^{i\vec{\lambda}.\vec{x}} d(M, \mathcal{E}(\lambda_0, \vec{\lambda})L)_{\mathcal{H}} ,$$

with $M \equiv M(1/2,0)$, $L \equiv L(1/2,0)$ and $\mathcal{E}(\lambda_0, \vec{\lambda})$ is the product for the spectral families of $\check{T}_0$ and $P^{i=1,2,3}$. By taking the Fourier transform, we can identify exactly the singularities of the Fourier transform of $\mathcal{G}$ with points in the E-M spectrum.

The rhs of Eq. (2) is valid for all vectors in $\mathcal{H}_e$. It turns out that, to reveal the appropriate combination of gauge invariant fields, we use the decoupling of hyperplane method (DHM). This method, besides revealing composite gauge-invariant fields, gives us good control properties of the decay of correlations and is a key result to show the upper gap property (i.e., that the low-lying E-M spectrum is generated by isolated dispersion curves).

In the DHM [4], in the action, we replace κ by $\kappa_p \in \mathbf{C}$ for bonds p connecting $u^0 + 1/2 \le p \le v^0 - 1/2$ and Taylor expand the general two-point correlation of Eq. (2) as a function of this new parameter. As shown in Ref. [3], calling the coefficient of κ_p^n at $\kappa_p = 0$ by $\mathcal{G}_{ML}^{(n)}$, the zeroth and first order terms give zero. For the second κ_p coefficient we get (summation over repeated indices is implied below), with $w = (p, \vec{w})$,

$$\mathcal{G}_{ML}^{(2)}(x,y) = \sum_{\vec{w}} \mathcal{G}^{(0)}_{M\bar{\mathcal{M}}_{\vec{\gamma}\vec{g}}}(x,w)\mathcal{G}^{(0)}_{\bar{\mathcal{M}}_{\vec{\gamma}\vec{g}}L}(w+e^0,y),$$

where we have used the shorthand notation

$$\bar{\mathcal{M}}_{\vec{\gamma}\vec{g}}(x) = \frac{1}{\sqrt{3}}\, \bar{\psi}_{a\gamma^\ell g_1}(x)\psi_{a\gamma^u g_2}(x) , \quad (3)$$

for $\vec{\gamma} \equiv (\gamma^\ell, \gamma^u)$ and $g \equiv (g_1, g_2)$, and where lower $\gamma^\ell = 3,4$ ($\gamma^u = 1,2$) is a lower (upper) spin index.

For closure, i.e. to get the same correlations on both the rhs and the lhs above, we choose $M \equiv \bar{\mathcal{M}}_{\vec{\alpha}\vec{f}}(x)$ and $L \equiv \bar{\mathcal{M}}_{\vec{\beta}\vec{h}}(x)$. This is how the local gauge-invariant mesons fields emerge naturally. No a priori use group theory is needed. Taking L and M as above we obtain the meson two-point correlation

$$\mathcal{G}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}(u,v) \equiv \begin{cases} \langle \mathcal{M}_\ell(u)\bar{\mathcal{M}}_{\ell'}(v)\rangle , & u^0 \le v^0 \\ \langle \bar{\mathcal{M}}_\ell(u)\mathcal{M}_{\ell'}(v)\rangle^* , & u^0 > v^0 , \end{cases}$$

where, for simplicity, we use the notation $\ell \equiv \vec{\alpha}\vec{f}$ and $\ell' \equiv \vec{\beta}\vec{g}$ in the expression above. The dimension of the matrix valued two-point correlation regarding (spin $\times$ isopin) is $(2\times 3)^2 = 36$.

To find the singularities of the Fourier transform of $\mathcal{G}_{\bar{\mathcal{M}}\bar{\mathcal{M}}}$, by a Neumann series, we define the convolution inverse to the two-point correlation, namely $\Lambda \equiv \mathcal{G}^{-1}$. More precisely, the reason behind the introduction of Λ is that we prove, using the DHM, that Λ decays faster than $\mathcal{G}$. Thus, its Fourier transform $\tilde{\Lambda}(p)$, $\tilde{\mathcal{G}}(p)\tilde{\Lambda}(p) = 1$, has a larger analyticity domain in p^0 than $\tilde{\mathcal{G}}(p)$. This domain turns out to be the strip $|\Im m\, p^0| < -(4-\epsilon)\ln\kappa$. Then, $\tilde{\Lambda}^{-1}(p) = [\mathrm{cof}\,\tilde{\Lambda}(p)]^t / \det[\tilde{\Lambda}(p)]$, provides a meromorphic extension of $\tilde{\mathcal{G}}(p)$. The singularities of $\tilde{\Lambda}^{-1}(p)$ are solutions $\omega(\vec{p})$ of the equation

$$\det[\tilde{\Lambda}(p^0 = iw(\vec{p}), \vec{p})] = 0 . \quad (4)$$

The solutions $w(\vec{p})$ correspond to the meson dispersion curves and the masses are $w(\vec{p} = \vec{0})$.

To identify the dispersion curves with the eightfold way mesons, we need to define spin. However, spin is *not* a symmetry of our model action. To define total spin and its $z-$component, we use improper zero momentum states. In this way, the total spin operator J and its $z-$component J_z are defined using $\pi/2$ rotations about the spatial coordinate axis and agree with the infinitesimal generators for the continuum (see Refs. [3,4]).

We now give a brief description of our main results. The existence of the mesons is manifested by isolated dispersion curves in the E-M spectrum up to near the two-meson threshold of $\approx -4\ln\kappa$. By considering spin, and the usual $SU(3)_f$ (total) isospin I, hypercharge Y and quadratic Casimir C_2, the 36 mesons can be grouped into four isospin nonets: the pseudoscalar and the vector mesons. There are 9 pseudoscalar mesons ($J = 0$) and 27 vector mesons

($J = 1$). Each nonet decomposes into an isospin singlet ($C_2 = 0$) and an octet ($C_2 = 3$). For $\beta = 0$, up to and including $\mathcal{O}(\kappa^4)$, they all have the same mass $M(\kappa) = -2\ln\kappa - 3\kappa^2/2 + \kappa^4 r(\kappa)$, with $r(\kappa)$ analytic and $r(0) \neq 0$. For $\beta \neq 0$, $M(\kappa,\beta) + 2\ln\kappa$ is jointly analytic in κ and β. For zero-momentum states, the masses are independent of J_z. Therefore, as for the pseudo-scalar mesons, all vector mesons have the same mass. For $\beta = 0$, we show that there is a mass splitting between the vector (v) and pseudo-scalar (p) mesons given by $[r_p(\kappa) - r_v(\kappa)]\kappa^4 = 2\kappa^4 + \mathcal{O}(\kappa^6)$; this splitting which persists for $\beta \neq 0$. For $\vec{p}_\ell^{\,2} \equiv \sum_{j=1,2,3} 2(1-\cos p^j)$, the meson dispersion curves are all of the form ($c = p, v$)

$$w_c(\kappa,\vec{p}) = -2\ln\kappa - \frac{3}{2}\kappa^2 + \frac{1}{4}\kappa^2\vec{p}_\ell^{\,2} + \kappa^4 r_c(\kappa,\vec{p})\,,$$

with $|r_c(\kappa,\vec{p})| \leq$ const and $r_p(\kappa,\vec{p})$ is jointly analytic in κ and p^j for $|\kappa|$, $|\Im m\, p^j|$ small. The various $w_v(\kappa,\vec{p})$ may depend on $|J_z|$.

These spectral results are obtained in the space $\mathcal{H}_{\bar{\mathcal{M}}} \subset \mathcal{H}_e$ generated by the fields of Eq. (3) using the Rouché and the analytic implicit function theorems to solve Eq. (4). As $M(\kappa) \approx -2\ln\kappa$ and $e^{ip^0x^0}$ gives negative powers of κ, to control $\tilde{\mathcal{G}}$ and $\tilde{\Lambda}$, up to order κ^4, we need to know the short-distance behavior of $\mathcal{G}$ and Λ up to order κ^{12}. Below, we sketch how to extend the spectral results from $\mathcal{H}_{\bar{\mathcal{M}}}$ to all $\mathcal{H}_e$. Combining this result with a similar spectral result for baryons proves confinement up to near the two-meson threshold.

4. Extension of the results to all $\mathcal{H}_e$

Following Refs. [3,4], a correlation subtraction method is used to show that the eightfold way meson spectrum is the only spectrum in all $\mathcal{H}_e$, up to near the two-meson threshold of $\approx -4\ln\kappa$. For this, we introduce the subtracted correlation

$$\mathcal{F} = \mathcal{G}_{LL} - \mathcal{G}_{L\bar{\mathcal{M}}}\Lambda\mathcal{G}_{\bar{\mathcal{M}}L}\,.$$

The general idea here is to show that the Fourier transform $\tilde{\mathcal{F}}$ of $\mathcal{F}$ has a larger region of analyticity than the Fourier transform of the two-meson correlation $\tilde{\mathcal{G}}_{\bar{\mathcal{M}}\bar{\mathcal{M}}} \equiv \tilde{\mathcal{G}}$. This follows using the DHM, to show that $\mathcal{F}(u,v)$ decays as $\kappa^{4|x^0|}$, for $x = v - u$. $\mathcal{G}_{L\bar{\mathcal{M}}}$ and $\mathcal{G}_{\bar{\mathcal{M}}L}$ have the same spectral representation as $\mathcal{G}$. The Fourier transform $\tilde{\Lambda}$ of Λ has a larger p^0 region of analyticity of $\approx -4\ln\kappa$. This shows that the singularities for imaginary momentum of the Fourier transform of $\mathcal{G}_{LL}$, for arbitrary $L \in \mathcal{H}_e$, coincides with those of $\tilde{\mathcal{G}}$. However, from section 3, the only possible singularities of the Fourier transform of $\mathcal{G}_{LL}$ coincide with the eightfold way mesons.

5. Final remarks

We have given a sketch of the main ideas to obtain our results validating the eightfold way classification scheme for mesons. The baryon eightfold way is similarly obtained. The eightfold way particles correspond to the only spectrum up to near the two-meson threshold. Our results are exact and appear in Refs. [3] and [4]. The control of the one-hadron spectrum, as done here, is a necessary step to go up in the spectrum, determine the two-hadron bound-state spectrum and search for exotic states as tetraquarks, pentaquarks and e.g. dibaryons. Our method is suitable for performing this analysis.

REFERENCES

1. E. Seiler, Lect. Notes in Phys. **159**, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Springer, New York, 1982.
2. M. Gell-Mann and Y. Ne'eman, *The Eightfold Way*, Benjamin, New York, 1964.
3. A. Francisco Neto, M. O'Carroll and P.A. Faria da Veiga, Phys. Rev. D77, 054503 (2008); J. Math. Phys. 49, 072301 (2008).
4. P.A. Faria da Veiga and M. O'Carroll, J. Math. Phys. 49, 042303 (2008). P.A. Faria da Veiga and M. O'Carroll, *Dynamical Eightfold Way in Strongly Coupled Lattice QCD* (2008), pre-print.
5. K. Wilson, in *New Phenomena in Subnuclear Physics*, Part A, A. Zichichi ed., Plenum Press, NY, 1977.
6. J.Hoek, N. Kawamoto, and J.Smit, Nucl. Phys. **B 199**, 495 (1982). J.Hoek and J.Smit, Nucl. Phys. **B 263**, 129 (1986).