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TECHNICAL NOTE

MATRIX FORMULATION OF THE DYNAMIC ANALYSIS OF SDOF SYSTEMS IN THE FREQUENCY DOMAIN

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Abstract—In this note a matrix formulation of the frequency domain dynamic analysis of SDOF systems is presented. The number of terms in the Discrete Fourier Transforms can be arbitrarily selected and the transforms are implicitly calculated in the procedure which leads to the response in the time domain.

INTRODUCTION

The analysis of dynamic response in frequency domain is strongly indicated for structural systems with frequency-dependent properties. Linear structural dynamic analysis in frequency domain is well known and had a great development with the use of the Fast Fourier Transform (FFT) algorithm. Nevertheless, only recently methods of non-linear dynamic structural analysis in the frequency domain have been developed. Kawamoto [1] presented an iterative method called Hybrid Frequency-Time Domain method, or HFTD method, in which the non-linearities are treated as pseudo-forces. Darbre and Wolf [2] presented the segmenting version of HFTD procedure and demonstrated its convergence criterion. Venancio Filho and Claret [3] presented a method for non-linear dynamic analysis in frequency domain based on a step-by-step incremental technique with linearized steps and a secant stiffness. All these methods use the FFT algorithm for calculation of direct and inverse discrete Fourier transforms.

Although the FFT algorithm is computationally very efficient it can be very hard to work in an iterative or step-by-step non-linear analysis. The number of terms in the discrete series, using the FFT algorithm, must be a power of two. Thus, if no sufficient precision is achieved with N terms, only $2N$ terms can be used. This fact means that the computational effort and the storage memory needed by the algorithm can increase very rapidly. On the other hand, the FFT algorithm is apart from the procedure of the response calculation in the frequency domain, implying in the repetition of a set of operations, every time it is called.

In this note a matrix formulation of the dynamic response analysis of SDOF systems in the frequency domain is presented in which the number of terms in the Discrete Fourier Transforms (DFTs) is arbitrarily selected. The only restriction is that it must be an odd integer. An important feature of this formulation is that the DFTs are implicitly executed in the same procedure that leads to the response in the time domain. It turns out that it is very suitable for non-linear dynamic analysis in the frequency domain.

CLASSICAL FORMULATION

The dynamic response of a SDOF system in the frequency domain can be expressed by the following equations [4]

$$v(t_n) = \frac{\Delta\bar{\omega}}{2\pi} \sum_{m=0}^{N-1} H(\bar{\omega}_m) P(\bar{\omega}_m)\, e^{i2\pi(mn/N)} \tag{1}$$

and

$$P(\bar{\omega}_m) = \Delta t \sum_{n=0}^{N-1} p(t_n)\, e^{-i2\pi(mn/N)}. \tag{2}$$

The total time interval T_p in which the response is to be calculated is divided into N (N odd) equal time intervals given by

$$\Delta t = \frac{T_p}{N} \tag{3}$$

and the discrete times in which the load is defined are given by

$$t_n = n\Delta t = n\frac{T_p}{N}, \quad (0 \leqslant n \leqslant N-1). \tag{4}$$

The frequency range is likewise divided into N equal intervals $\Delta\bar{\omega}$ expressed as

$$\Delta\bar{\omega} = \frac{2\pi}{T_p} \tag{5}$$

and the discrete frequencies $\bar{\omega}_m$ are taken according Table A1 (see Appendix 1).

In eqn (2), $P(\bar{\omega}_m)$ is discrete Fourier transform of the load; in eqn (1), $H(\bar{\omega}_m)\ P(\bar{\omega}_m)$ is the discrete Fourier transform of the response (or the response in the frequency domain) and $v(t_n)$ is the inverse discrete Fourier transform of the response (or the response in the time domain).

The dynamic response expressed by eqns (1) and (2) can be numerically determined by the FFT algorithm. In dealing with this algorithm, N must be a power of two and, consequently, an even integer. Nevertheless, as will be subsequently shown, when N is even there is an imaginary term in the response. In order to get rid of this term, N must be odd.

MATRIX FORMULATION

Let

$$\mathbf{p} = \{p(t_0), p(t_1), p(t_2), \dots, p(t_n), \dots, p(t_{N-1})\} \quad (6)$$

and

$$\mathbf{v} = \{v(t_0), v(t_1), v(t_2), \dots, v(t_n), \dots, v(t_{N-1})\} \quad (7)$$

be, respectively, the vectors of the load and the response at the discrete times

$$t_n = n\Delta t, \quad (n = 0, 1, 2, \dots, N-1), \quad (8)$$

and let

$$\mathbf{P} = \{P(\bar{\omega}_0), P(\bar{\omega}_1), P(\bar{\omega}_2), \dots, P(\bar{\omega}_m), \dots, P(\bar{\omega}_{N-1})\} \quad (9)$$

be the vector of the discrete Fourier transform of the load defined at the discrete frequencies $\bar{\omega}_m$ interpreted according to Table A1.

With the definition of eqns (6) and (9), eqn (2) can be cast in matrix form as

$$\mathbf{P} = \Delta t\,\mathbf{E^*P}, \quad (10)$$

where the $(N \times N)$ matrix $\mathbf{E^*}$ is defined as the matrix whose generic term E^*_{mn} is

$$E^*_{mn} = e^{-imn(2\pi/N)} \quad (11)$$

or, explicitly

$$\mathbf{E^*} = \begin{bmatrix} e^0 & e^0 & e^0 & \cdots e^0 & \cdots e^0 \\ & e^{-i(2\pi/N)} & e^{-i2(2\pi/N)} & \cdots e^{-in(2\pi/N)} & \cdots e^{-i(N-1)(2\pi/N)} \\ & & e^{-i4(2\pi/N)} & \cdots e^{-i2n(2\pi/N)} & \cdots e^{-i2(N-1)(2\pi/N)} \\ & & & \vdots & \\ & & & e^{-imn(2\pi/N)} & e^{-im(N-1)(2\pi/N)} \\ & \text{Symmetric} & & & \vdots \\ & & & & e^{-i(N-1)^2(2\pi/N)} \end{bmatrix}. \quad (12)$$

By the same token, the response from eqn (2) is written in matrix form as

$$\mathbf{v} = \frac{\Delta\bar{\omega}}{2\pi}\,\mathbf{EHP}, \quad (13)$$

where $\mathbf{E}$ is the matrix defined in eqn (11) with positive signs in the exponentials instead of negative ones, and $\mathbf{H}$ is the diagonal matrix formed with the complex frequency response functions calculated at the discrete frequencies of Table A1. The typical term of $\mathbf{H}$ is given by

$$H(\bar{\omega}_m) = (k - m\bar{\omega}_m^2 + i\bar{\omega}_m c)^{-1}, \quad (0 \leqslant m \leqslant N-1), \quad (14)$$

where k, m, and c are the stiffness, mass, and damping of the SDOF system, respectively. Substituting now $\mathbf{P}$ from eqn (10) into eqn (13), the following equation is obtained

$$\mathbf{v} = \frac{1}{N}\,\mathbf{EHE^*p}. \quad (15)$$

Equation (15) expresses the matrix formulation of the dynamic analysis of SDOF systems in the frequency domain. In the sequel, it is proven that the matrix product $\mathbf{EHE^*p}$ is real, provided N is odd.

Consider from eqn (2) $P(\bar{\omega}_m)$ and $P(\bar{\omega}_{N-m})$, with $m = 0, 1, \dots, (N-1)/2$ written in indicial notation, respectively, as

$$\frac{1}{\Delta t} P(\bar{\omega}_m) = E^*_{mj} p_j \quad (16)$$

and

$$\frac{1}{\Delta t} P(\bar{\omega}_{N-m}) = E^*_{(N-m)j} p_j, \quad (17)$$

where $j = 1, 2, \dots, N-1$. All the corresponding terms in the summations of eqns (16) and (17) are complex conjugates (except the first ones which are real) in face of the proof given in Appendix 2. Therefore, $P(\bar{\omega}_m)$ and $P(\bar{\omega}_{N-m})$ are complex conjugates.

Consider now eqn (13) written as

$$\frac{2\pi}{\Delta\bar{\omega}}\,\mathbf{v} = \mathbf{AP}, \quad (18)$$

where A is given by

$$\mathbf{A} = \mathbf{EH}. \tag{19}$$

The nth line of matrix $\mathbf{A}$ from eqn (19) can be written as the vector

$$\mathbf{A}_n = \left\{ \frac{1}{k} \cdots e^{inm(2\pi/N)} H(\bar{\omega}_m) \cdots e^{in(N-m)(2\pi/N)} \times H(\bar{\omega}_{N-m}) \cdots \right\} \tag{20}$$

As $(e^{inm(2\pi/N)}, e^{in(N-m)(2\pi/N)})$ and $[H(\bar{\omega}_m), H(\bar{\omega}_{N-m})]$ are pairs of complex conjugates (see Appendices 2 and 1, respectively), the typical pair displayed in eqn (20) is also a pair of complex conjugates.

Multiply now $\mathbf{A}_n$, from eqn (20), by $\mathbf{P}$, from eqn (9), in order to obtain the typical term of $\mathbf{v}$, v_n, from eqn (18). The result, in indicial notation, is

$$\frac{\Delta\bar{\omega}}{2\pi} v_n = E_{nm} H(\bar{\omega}_m) P(\bar{\omega}_m). \tag{21}$$

In this summation, all pairs like

$$[E_{nm} H(\bar{\omega}_m) P(\bar{\omega}_m), E_{n(N-m)} H(\bar{\omega}_{N-m}) P(\bar{\omega}_{N-m})], \tag{22}$$

where $m = 1, 2, \ldots, N-1$ and where $\bar{\omega}_m$ is interpreted according to Table A1, are complex conjugates pairs. On the other hand the first term ($m = 0$) in the summation of eqn (21) is real. Therefore, v_n is a sum of a real term (the first one) with pairs of complex conjugates which finally proves that v_n is real and, consequently, $\mathbf{v}$ is real.

A very important point that must be emphasized is that N must be odd, otherwise there would be in the summation of eqn (21), a central term

$$E_{n(N/2)} H(\bar{\omega}_{N/2}) P(\bar{\omega}_{N/2})$$

which has not its complex conjugate in order to form a complex conjugate pair. In this way, there would exist an imaginary term in the summation which produces v_n. This is contrary to the condition implied in FFT algorithm that N should be a power of two and, consequently, even.

FINAL REMARKS

Equation (15) establishes a matrix formula for the analysis of the response of a SDOF system in the frequency domain. The aim of this formulation is to minimize the computational effort required in non-linear analysis in the frequency domain. On the other hand, in eqn (15) the only condition implied over N is that it should be odd. Thus, N can be changed freely to achieve sufficient precision in the response, optimizing the space of memory in the computer and the number of operations in the process.

The matrices $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{E}^*$ have properties that can simplify the numerical implementation of eqn (15), also reducing the computational effort.

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APPENDIX 1

The discrete frequencies employed in this formulation must be interpreted according Table A1. Taking into account the frequencies $\bar{\omega}_m$ from Table A1, $H(\bar{\omega}_m)$ and $H(\bar{\omega}_{N-m})$, eqn (14), are complex conjugate.

Table A1. Discrete frequencies (N odd)

m	m or $(N-m)$	$\bar{\omega}_m$
0	0	0
1	1	$\Delta\bar{\omega}$
2	2	$2\Delta\bar{\omega}$
...	...	...
$(N-1)/2$	$(N-1)/2$	$[(N-1)/2]\Delta\bar{\omega}$
$(N+1)/2$	$(N-1)/2$	$[-(N-1)/2]\Delta\bar{\omega}$
...	...	...
$N-2$	2	$-2\Delta\bar{\omega}$
$N-1$	1	$-\Delta\bar{\omega}$

APPENDIX 2

Apply Euler formula to $e^{imn(2\pi/N)}$ in order to have

$$e^{imn(2\pi/N)} = \cos\left(mn\frac{2\pi}{N}\right) + i\sin\left(mn\frac{2\pi}{N}\right) \tag{A2.1}$$

and to $e^{(i(N-m)n(2\pi/N)}$ to have

$$\begin{aligned} e^{i(N-m)n(2\pi/N)} &= \cos\left[(N-m)n\frac{2\pi}{N}\right] + i\sin\left[(N-m)n\frac{2\pi}{N}\right] \\ &= \cos(2\pi n)\cos\left(mn\frac{2\pi}{N}\right) + \sin(2\pi n)\sin\left(mn\frac{2\pi}{N}\right) \\ &\quad + i\left[\sin(2\pi n)\cos\left(mn\frac{2\pi}{N}\right) - i\sin\left(mn\frac{2\pi}{N}\right)\cos(2\pi n)\right] \\ &= \cos\left(mn\frac{2\pi}{N}\right) - i\sin\left(mn\frac{2\pi}{N}\right). \end{aligned} \tag{A2.2}$$

Equations (A2.1) and (A2.2) prove that $e^{imn(2\pi/N)}$ and $e^{i(N-m)n(2\pi/N)}$ are complex conjugates.