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TECHNICAL NOTE

FREQUENCY DOMAIN DYNAMIC ANALYSIS OF MDOF SYSTEMS: NODAL AND MODAL COORDINATES FORMULATIONS

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Abstract—General formulations for the frequency domain dynamic analysis of MDOF systems are presented in nodal and modal coordinates. The definition of damping is very broad and includes non-proportional and hysteretic damping.

INTRODUCTION

The analysis of dynamic response in frequency domain is strongly indicated for structural systems with frequency-dependent properties and hysteretic damping. In a previous Technical Note, Venancio-Filho and Claret [1] presented the matrix formulation of the dynamic analysis of SDOF systems in the frequency domain. In this formulation the DFTs (Discrete Fourier Transforms) are implicitly executed in the same procedure that leads to the response in the time domain. This is the concept of IFT (Implicit Fourier Transform). In the present Note the formulation for MDOF is developed in nodal and modal coordinates. The consideration of damping is the most general, encompassing non-proportional and hysteretic damping. In the modal coordinate formulation an efficient iterative procedure based on the pseudo-force concept can be used to obtain the complex frequency response matrix.

FORMULATION IN NODAL COORDINATES

Consider a MDOF system with J nodal coordinates. The dynamic equilibrium equation in the frequency domain is

$$[-\omega^2[m] + i(\omega[c] + [k_D]) + [k]]\{v_0\} = \{p_0\}. \tag{1}$$

In this equation $[m]$ and $[k]$ are, respectively, the $(J \times J)$ mass and stiffness matrices; $[c]$ is the $(J \times J)$ viscous damping matrix and $[k_D]$ is the $(J \times J)$ hysteretic damping matrix. This matrix is built up by adequate superposition of the element stiffness matrices multiplied by the respective damping coefficients. $\{v_0\}$ and $\{p_0\}$ are, respectively, the $(J \times 1)$ vectors of the amplitudes of the nodal displacements and the load at frequency ω.

From eqn (1) one obtains by inversion:

$$\{v_0\} = [H(\omega)]\{p_0\}, \tag{2}$$

where

$$[H(\omega) = [-\omega^2[m] + i(\omega[c] + [k_D]) + [k]]^{-1} \tag{3}$$

is the complex frequency response matrix. The dependency of $[H(\omega)]$ from ω stems not only from ω and ω^2 but also from $[c]$ and $[k]$ when they are frequency dependent.

For an arbitrary loading $\{p(t)\}$, the nodal displacement response in the frequency domain is given by [2]:

$$\{V(\omega)\} = [H(\omega)]\{P(\omega)\}, \tag{4}$$

where $\{P(\omega)\}$ and $\{V(\omega)\}$ are, respectively, the loading and nodal displacements FTs (Fourier Transforms). The DFT (Discrete Fourier Transform) of $\{p(t)\}$ is $\{P(\omega)\}$, whose elements are $P(\omega_m)$, given by Clough and Penzien [2],

$$P(\omega_m) = \Delta t \sum_{n=0}^{N-1} p(t_n) e^{-i2\pi(mn/N)} \tag{5}$$

where ω_m $(n = 0, 1, 2, \ldots, N-1)$ and t_n $(n = 0, 1, 2, \ldots, N-1)$ are, respectively, the discrete frequencies and times, and N is the number of points in the DFT. Taking into account eqns (4) and (5), the DFT of the nodal displacement v_i is

$$V_i(\omega_m) = \Delta t \sum_{j=1}^{J} H_{ij}(\omega_m) \sum_{n=0}^{N-1} p_j(t_n) e^{-i2\pi(mn/N)}. \tag{6}$$

The time domain response of the ith DOF, $v_i(t_n)$, is the IDFT (Inverse Discrete Fourier Transform) of $V_i(\omega_m)$, given by

$$v_i(t_n) = \frac{\Delta\omega}{2\pi} \sum_{m=0}^{N-1} V_i(\omega_m) e^{i2\pi(mn/N)}. \tag{7}$$

Substituting now into eqn (7) $V_i(\omega_m)$ from eqn (6), the final following result is obtained:

$$v_i(t_n) = \frac{1}{N} \left\{ \sum_{m=0}^{N-1} e^{i2\pi(mn/N)} \times \left[\sum_{j=1}^{J} H_{ij}(\omega_m) \left(\sum_{n=0}^{N-1} p_j(t_n) e^{-i2\pi(mn/N)} \right) \right] \right\} \tag{8}$$

noting that

$$\frac{\Delta\omega\Delta t}{2\pi} = \frac{1}{N}. \quad (9)$$

FORMULATION IN MODAL COORDINATES

The modal transformation

$$\{v_0\} = [\Phi]\{y_0\} \quad (10)$$

is now considered. $[\Phi]$ is the $(J \times K)$ $(K \ll J)$ matrix of normal modes and $\{y_0\}$ is the $(K \times 1)$ vector of modal coordinates. Introducing eqn (10) into eqn (1) and pre-multiplying by $[\Phi]^T$, one obtains

$$[-\omega^2[I] + i(\omega[C] + [K_D]) + [\Omega^2]]\{y_0\} = \{P_0\}. \quad (11)$$

The normal modes are normalized in such a way that

$$[\Phi]^T[m][\Phi] = [I] \quad (12)$$

where $[I]$ is the $(K \times K)$ unit matrix. In this way

$$[\Phi]^T[k][\Phi] = [\Omega^2], \quad (13)$$

where $[\Omega^2]$ is a $(K \times K)$ diagonal matrix containing the K natural frequencies squared. Moreover, in eqn (11),

$$[C] = [\Phi]^T[c][\Phi], \quad (14a)$$

$$[K_D] = [\Phi]^T[k_D][\Phi], \quad (14b)$$

and

$$[P_0] = [\Phi]^T\{p_0\}. \quad (15)$$

The inversion of eqn (11) produces the following result:

$$\{y_0\} = [\bar{H}(\omega)]\{P_0\}, \quad (16)$$

where

$$[\bar{H}(\omega)] = [-\omega^2[I] + i(\omega[C] + [K_D]) + [\Omega^2]]^{-1} \quad (17)$$

is the complex frequency response matrix in modal coordinates.

Starting from eqn (16) and following in modal coordinates the steps of eqns (4)–(8), the final result for the ith modal coordinate is obtained as

$$v_i(t_n) = \frac{1}{N}\sum_{m=0}^{N-1} e^{i2\pi(mn/N)} \times \left[\sum_{k=1}^{K} \bar{H}_{ik}(\omega_m)\left(\sum_{n=0}^{N-1} P_k(t_n)e^{-i2\pi(mn/N)}\right)\right]. \quad (18)$$

If there is no hysteretic damping and the MDOF system is classically damped (or damping is proportional) then matrix $[C]$ in eqn (14a) is diagonal. Consequently, the matrix in eqn (11) is diagonal, the inversion is trivial, and $[H(\omega)]$, eqn (17), is a diagonal matrix. In the general case, the matrix in eqn (11) is not diagonal.

Recently Jangid and Datta [3] proposed an iterative procedure, based on the pseudo-force concept introduced by Claret and Venancio-Filho [4] for time history analysis, to obtain $[\bar{H}(\omega)]$ in the general case, which is computationally efficient and attractive. Summarizing, this procedure is as follows. Consider the equilibrium equation in modal coordinates for harmonic excitation,

$$\{\ddot{y}\} + [C]\{\dot{y}\} + i[K_D]\{y\} + [K]\{y\} = \{P_0\}e^{i\omega t}. \quad (19)$$

Matrices $[C]$ and $[K_D]$ are split as

$$[C] = [C_D] + [C_f] \quad (20)$$

and

$$[K_D] = [K_{Dd}] + [K_{Df}]. \quad (21)$$

In eqn (20) $[C_d]$ is a diagonal matrix which has the diagonal elements of $[C]$, and $[C_f]$ has zero diagonal elements and the corresponding off-diagonal elements of $[C]$. The same follows for $[K_D]$ in eqn (21). Substituting for $[C]$ and $[K_D]$ in eqn (19) from eqns (20) and (21), respectively, the following equation is obtained:

$$\{\ddot{y}\} + [C_d]\{\dot{y}\} + i[K_{Dd}]\{y\} + [K]\{y\} = \{P_0\}e^{i\omega t} - [C_f]\{\dot{y}\} - i[K_{Df}]\{y\}. \quad (22)$$

Introducing into eqn (22) the steady-state solution $\{y\} = \{y_0\}e^{i\omega t}$ this equation is transformed, after inversion, into

$$\{y_0\} = [\bar{H}_d(\omega)]\left[\{P_0\} - (i\omega[C_f] + i[K_{Df}])\{y_0\}\right] \quad (23)$$

where

$$[\bar{H}_d(\omega) = [-\omega^2[I] + i\omega[C_d] + i[K_{Dd}] + [K]]^{-1} \quad (24)$$

is a diagonal matrix and noting that inversion is trivial. The term $(-i\omega[C_f] + i[K_{Df}])\{y_0\}$ is a pseudo-force term which takes into account the general form of the damping matrices.

Equation (23) is solved by an iterative procedure, the kth step of which is

$$\{y_0^{(k)}\} = [\bar{H}_d(\omega)]\left[\{P_0\} - (i\omega[C_f] + i[K_{Df}])\{y_0^{(k-1)}\}\right]. \quad (25)$$

In the first step

$$-(i\omega[C_f] + i[K_{Df}])\{y_0^{(0)}\} = 0. \quad (26)$$

The final result of the iterative steps is

$$[\bar{H}^{(k)}(\omega)] = [\bar{H}_d(\omega)]\left[[I] - [Z] + [Z]^2 - [Z]^3 + \cdots + (-1)^{(k-1)}[Z]^{(k-1)}\right] \quad (27)$$

where

$$[Z] = (i\omega[C_f] + i[K_{Df}])[\bar{H}_d(\omega)]. \quad (28)$$

The geometric series in eqn (27) can be expressed as

$$[\bar{H}^{(k)}(\omega)] = [\bar{H}_d(\omega)]\frac{[[I] + (-1)^{(k-1)}[Z]^{(k-1)}]}{[[I] + [Z]]}. \quad (29)$$

$[\bar{H}^{(k)}(\omega)]$ from eqns (27) or (29) is the complex frequency response matrix in modal coordinates obtained by the proposed iterative pseudo-force method.

For a convergent series, from eqn (29),

$$\lim_{k\to\infty}[\bar{H}^{(k)}(\omega)] = \frac{[\bar{H}_d(\omega)]}{[[I] + [Z]]} = [\bar{H}(\omega)] \quad (30)$$

where $[\bar{H}(\omega)]$ is the matrix defined in eqn (17).

The iterative procedure is convergent if $|[Z]| < 1$ where $|[Z]|$ is the following norm of $[Z]$:

$$[Z] = \{\det[[Z][Z]^{T*}]\}^{1/2} \quad (31)$$

where * denotes complex conjugate.

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