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# COMPARING CURVATURE ESTIMATION TECHNIQUES

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**Abstract** This article presents a careful comparative evaluation of two techniques for numerical curvature estimation of 2D closed contours (more specifically closed, regular and simple parametric curves). The considered methods are: (a) a 1-D Fourier-based approach; and (b) a 2-D Fourier-based approach involving the embedding of the contour into a 2-D regular surface (presented for the first time in this article). Both these techniques employ Gaussian smoothing as a regularizing condition in order to estimate the first and second derivatives needed for curvature estimation. These methods are considered according to a multiresolution approach, where the standard deviation of the Gaussians are used as scale parameters. The methods are applied to a standard set of curves whose analytical curvatures are known in order to estimate and compare the errors of the numerical approaches. Three kinds of parametric curves are considered: (i) curves with analytical description; (ii) curves synthesized in terms of Fourier components of curvature; and (iii) curves obtained by splines. A precise comparison methodology is devised which includes the adoption of a common spatial quantization approach (namely square box quantization) and the explicit consideration of the influence of the related smoothing parameters. The obtained results indicate that the 1-D approach is not only faster, but also more accurate. However, the 2-D approach is still interesting and reasonably accurate for applications in situations where the curvature along the whole 2-D domains is needed.

**Key-words:** Differential geometry, numerical techniques, curvature estimation, performance evaluation, Fourier transform.

## 1 INTRODUCTION

One of the keys to characterize and analyze visual information, as well as many natural signals, consists in removing the many

redundancies often found in such data [Barlow, 1994]. Typical images in our visual world are indeed characterized by a high degree of correlation between neighboring points. For instance, the current page is characterized by a high degree of correlation regarding both the letters (black points) and background (white points). Consequently, a first sensible step in addressing image analysis consists in applying some high-pass filter capable of enhancing high contrast points, i.e. the object borders, in detriment of low contrast regions. When such enhanced images are followed by binary border extraction, the resulting contours will often preserve the majority of the original visual information, reinforcing the importance of contours as compact and information preserving representations of visual shape. However, despite the redundancy removal implemented by border extraction, the obtained contours still may exhibit a high degree of redundancy. For instance, straight lines are maximally correlated 1-D elements in space, since they can be represented in terms of just their extremities. It becomes clear that additional levels of redundancy removal are needed in order to better represent visual shapes. Interestingly, the compaction allowed by redundancy removal is essential not only for reducing storage, but also allows the more relevant and salient information (such as vertices) to be emphasized. While the above mentioned piecewise linear scheme can be considered, it represents a first order approach which is not particularly effective for more elaborated curves. Indeed, piecewise linear representations can be understood as being dual to the analysis of high curvature points, since straight lines present the lowest curvature magnitude (i.e. zero). One particularly effective means of characterizing and representing shapes is in terms of *point curvature*, given by Equation (1) for parametric representation, which is not restricted to arc-length parameter.

$$k(t) = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{3/2}} \quad (1)$$

$$(x(t), y(t)) = \left( \int_0^t \cos(\theta(s)) ds + a, \int_0^t \sin(\theta(s)) ds + b \right) \quad (2)$$

where  $\theta(s) = \int_0^s k(w)dw$ , where both  $s$  and  $w$  are arc-lengths.

The special relevance of curvature as a shape descriptor stems from the following facts: (i) the curvature is conceptually and physically meaningful, indicating how bent a portion of a curve is; (ii) the curvature representation preserves information in the sense that the original curve can be recovered from its point curvatures, except for translation and rotation — see Equation (2); and (iii) as will be further discussed in this article, curvature expresses the inverse of the local spatial scale (related to the curvature center), thus providing a valuable indication of this important parameter. Indeed, the importance of digital curvature estimation has motivated a series of related approaches (e.g. [Mokhtarian & Mackworth, 1992; Medioni & Yasumoto, 1987; Baroni & Barletta, 1992]), as well as some comparative assessments [Worring & Smeulders, 1992; Fairney & Fairney, 1994]. As a matter of fact, curvature is one of the most important information about shape contours, being used in many different situations, such as for object classification or polygonal approximation [Pernus et al, 1993; Medioni & Yasumoto]. Nevertheless, in spite of its importance, no definitive numerical technique for curvature estimation has been obtained and vision researchers face are confronted with a multitude of different techniques for estimating the curvature of digital contours. While the curvature of continuous curves can often be easily and precisely determined by using the respective symbolic expression and derivatives involved in Equation 1, the problem of estimating curvature of spatially sampled contours is not straightforward. The principal problem with such "digital curvature" estimation approaches is that spatially sampled curves do not even present curvature in a strict sense, for they are no more than a set of singularities (isolated points). Thus, some regularizing pre-processing, such as smoothed interpolation, is needed before curvature can be estimated. In order to circumvent the problem of numerical differentiation, some works have developed techniques that estimate alternative measures, such as the  $c$ -curvature [Davis, 1977]. On the other hand, curvature may be directly estimated by using numerical differentiation (such as finite differences), interpolation [Medioni & Yasumoto, 1987] or signal processing techniques (such as convolution with differentiation kernels or Fourier properties) [Mokhtarian & Mackworth, 1992, Cesar & Costa, 1997]. Finally, there are methods that attempt to estimate the curvature directly from the 2-D data [Chen *et al.*, 1995].

The work reported in the current article considers the following two principal numerical approaches to the estimation of digital curvature of closed, regular, and simple parametrized curves: (a) a 1-D Fourier-based including Gaussian smoothing [Cesar and Costa, 1997]; and (b) a 2-D Fourier-based approach, involving 2-D embedding of the curve into a surface and Gaussian smoothing, presented for the first time in this article. These two approaches present similarities and differences. To start with, both techniques have a multiresolution nature defined by the standard deviation of the Gaussians, which are used to provide the regularization (under the constrain of smoothness) required for effective curvature estimation. In both cases, this regularization implements some interpolation scheme along the original isolated points in the contour. Yet, while the 1-D approach works directly on the 1-D parametrized representation of the curve, the 2-D method applies 2-D derivative filters and required the 1-D curve to be transformed into a surface. Such Fourier-based approaches use the well-

known derivative property of the Fourier transform [Castleman, 1996], given by Equation (3), where  $s(t)$  is the original signal in the time ( $t$ ) domain,  $a$  is the order of the derivative (a real value),  $j$  is the imaginary number,  $f$  stands for frequency, and  $\mathfrak{F}$  is the Fourier transform.

$$s^{(a)}(t) = \mathfrak{F}^{-1} \left\{ (2\pi j f)^a \mathfrak{F} \{s(t)\} \right\} \quad (3)$$

$$\frac{\partial^a \partial^b \phi(x, y)}{\partial x^a \partial y^b} = \phi_{xy}^{(a)(b)} = \mathfrak{F}^{-1} \left\{ (2\pi j u)^a (2\pi j v)^b \mathfrak{F} \{ \phi(x, y) \} \right\} \quad (4)$$

Although the accuracy and performance of the 1-D Fourier-based approach has been assessed previously by comparing the exact and estimated values for curves generated by splines [Mortenson, 1985], having produced encouraging results, it would be interesting to develop a more comprehensive and formal comparison approach. The main objective of the present article is not only to pursue such a possibility, but also to compare the obtained results with those produced by the considered 2-D Fourier-based scheme. As in [Cesar and Costa, 1997], curves with known analytical curvature are adopted as standard for comparison, using Euclidean metric. The considered curves, all closed regular and simple, include two analytical curves, a Fourier synthesized contour [Zahn and Roskies, 1972], and a curve obtained by using splines. Special attention is drawn to devising a fair comparison scheme. This includes a histogram-based limitation of curvature values in order to enforce a reasonable representation of curvature values in the orthogonal grid. In addition, the sensitivity of the two techniques with respect to the scale space parameter is explicitly and carefully considered. Firstly investigated is the effect of the scale over the estimation, and the best scales are then considered for all subsequent comparisons, i.e. the latter consider the best overall results for each curve with respect to each scale parameters. The obtained results are statistically analyzed, including errors distribution in terms of curvature ranges and dependence with the scale space parameters. In addition to accuracy, the execution speed is also considered and comparatively quantified.

The current article starts by reviewing the two considered numerical techniques for curvature estimation and proceeds by discussing the developed approach for comparing them. The obtained results are then presented and discussed.

## 2 1-D FOURIER-BASED CURVATURE ESTIMATION

The method presented in this section is based on the multiscale curvature approach (the *curvegram*) introduced in (Cesar & Costa (1997)). This approach has proven to be a useful practical tool in many situations, such as in stereo vision and in the analysis of neural images for biomedical applications.. We start with the parametric curve  $c(n) = (x(n), y(n))$ , where the parameter  $n = 0, \dots, N-1$ , where  $N$  is the number of points along the contour. This contour can be represented by a complex signal  $u(n) = x(n) + i y(n)$ , where  $i = \sqrt{-1}$ . The method makes use of the Fourier transform pair defined by  $u(n)$ , which can be defined as (Brigham, 1988):

$$U(s) = F\{u(n)\} = \sum_{n=1}^N u(n) e^{-i2\pi \frac{sn}{N}}, s = 0, \dots, N-1$$

$$u(n) = F^{-1}\{U(s)\} = \sum_{s=0}^{N-1} U(s) e^{i2\pi \frac{sn}{N}}$$

The curvature calculation involves the estimation of the discrete derivatives of the digital signal  $u(n)$ , which is done based on the Fourier transform and its derivative property (Brigham, 1988). The application of this property to the coefficients generated by FFT algorithms (which are usually applied to compute Fourier transforms) requires the definition of an auxiliary function  $\eta(s)$ :

$$\eta(s) = \begin{cases} s, & \text{if } s = 0, 1, \dots, N - N2 - 1 \\ N - s, & \text{if } s = N - N2, \dots, N - 1 \end{cases}$$

where  $N2 = \text{floor}(N/2)$  is the traditional truncation function. Function  $\eta(s)$  is used to cope with the standard indexing representation of the FFT. The first and the second derivatives of  $u(n)$  can then be defined as:

$$\dot{u}(n) = F^{-1}\{\dot{U}(s)\} = F^{-1}\{i 2\pi \eta(s) U(s)\} \quad (4)$$

$$\ddot{u}(n) = F^{-1}\{\ddot{U}(s)\} = F^{-1}\{-(2\pi \eta(s))^2 U(s)\} \quad (5)$$

where  $n = 0, 1, \dots, N-1$  and  $s = 0, 1, \dots, N-1$ . The multiscale behavior is introduced by filtering the original signal with a Gaussian  $G_a(s) = \exp(-a \eta(s)^2)$ :

$$u(n, a) = \Delta t F^{-1}\{U(s) G_a(s)\}$$

which is equivalent to a convolution (Brigham, 1988) with inversed Fourier transform of the above Gaussian function. The scaling parameter " $\Delta t$ " corresponding to the parameter quantization, presented for the first time in this article, is needed in order to conserve the center of mass of the closed curve and to altogether avoid the somewhat subjective normalization procedure, for instance in terms of perimeter or signal energy, used in (Cesar & Costa (1997)). The multiscale curvature description of the contour  $c(n)$ , which constitute the *curvegram* (Cesar & Costa (1997)), is defined as:

$$k(n, a) = \frac{-\text{Im}\{\dot{u}(n, a) \ddot{u}^*(n, a)\}}{|\dot{u}(n, a)|^3}$$

where  $z^*$  denotes complex conjugate.

### 3 2-D FOURIER-BASED CURVATURE ESTIMATION

While the technique discussed in the last section is based on 1-D parametrized complex contours, it is also possible to use 2-D differential operators such as that in Equation (6) to estimate the curvature. Given a regular and simple parametrized curve  $c(t)$  as in the previous section, they have to be extended onto

the 2-D domain in some way before such 2-D operators can be applied. Two possible alternatives for such an embedding are: (i) to fill the interior of  $c(t)$  with '1s'; and (ii) to apply a signed distance transform to the contour, in such a way that the interior becomes negative (resp. positive) and the exterior positive (resp. negative). The present article has adopted the former scheme. Once such an extension  $\phi(x, y)$  is achieved, the curvature of the contour defined by  $\phi(x, y) = a$  (a level-curve) can be estimated by using Equation (6).

$$k = \nabla \cdot \frac{\nabla \phi}{\|\nabla \phi\|} = \frac{\phi_{xx}\phi_y^2 - 2\phi_x\phi_{xy}\phi_{xy} + \phi_{yy}\phi_x^2}{(\phi_x^2 + \phi_y^2)^{3/2}} \quad (6)$$

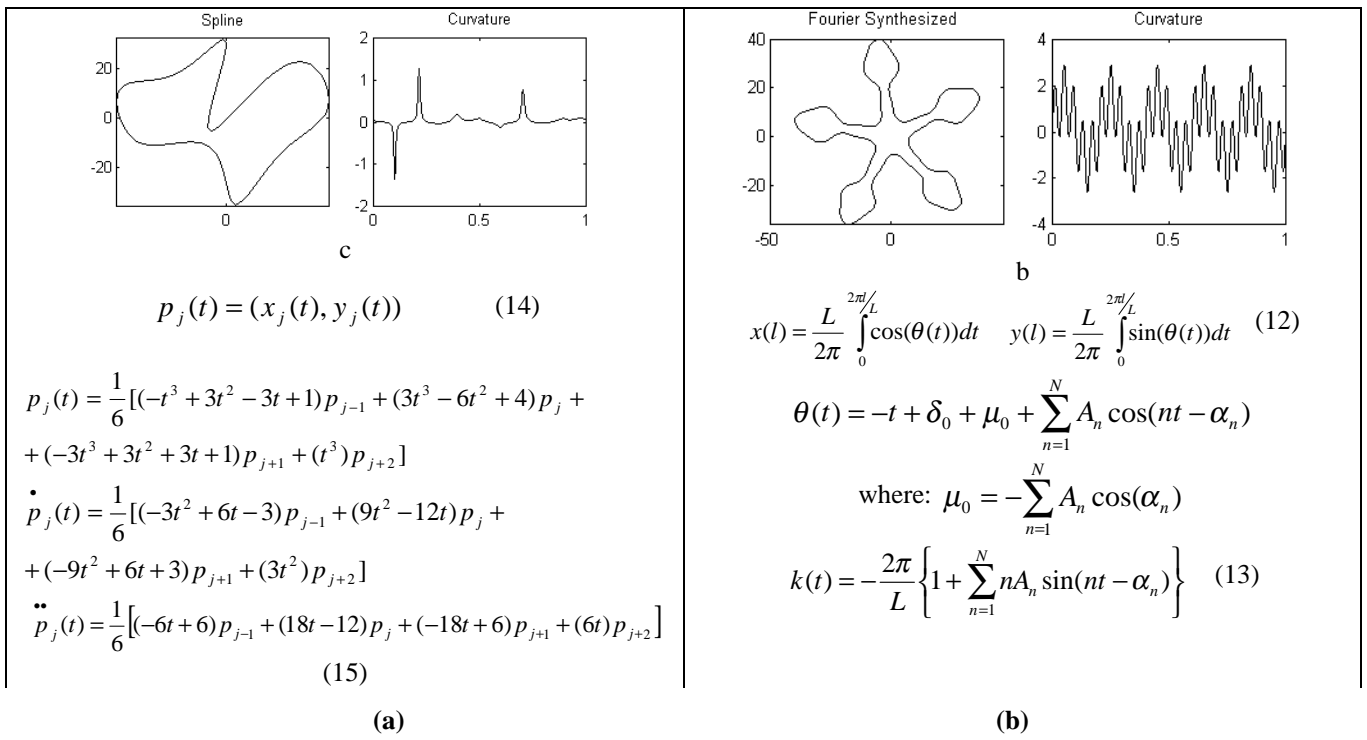
As with the 1-D approach described in the previous section, it is necessary to regularize  $\phi(x, y)$ , since this is represented in a spatially sampled space. This will be done by convolving  $\phi(x, y)$  with a circularly symmetric 2-D Gaussian given by Equation (7).

$$G_\sigma(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}} \quad (7)$$

### 4 THE COMPARISON METHODOLOGY

As reviewed in the previous sections, both Fourier-based methods can operate on the original contour and produce, for a specific spatial scale parameter  $\sigma$ , a numeric estimation of the point curvature along the closed curve. The main purpose of the current article is to provide a careful assessment of the performance of both techniques by taking into account accuracy, sensibility to parameter variation, and execution time. However, in order that meaningful conclusions can be reached, every care is needed in devising a reasonably comprehensive and fair comparison procedure.

The first important issue regards what curves to consider. Such curves should not only allow accurate analytic point curvature estimation, but also reasonably represent typical shapes found in image analysis and vision. While it is impossible to consider every class of shapes in an experimental assessment, we have considered three classes of curves, two of which re illustrated, jointly with their curvature plot and respective equations, in Figure 1. These include (i) analytical curves; (ii) curves synthesized in terms of Fourier components of curvature (Zahn & Roskies (1972)); and (iii) general curves generated by using splines (Fig. 1c) (Mortenson, 1985). The spline-baed contours address the curvature estimation of free forms, which represent in a more legitimate way those commonly found in real pictures. Within this framework, contours are generated interactively through the definition of a series of control points along the Cartesian plan. Once the B-splines are expressed in terms of analytical polynomial expressions, it is possible to calculate the curvature of each point along the contour. It should be observed that while the equations in Figure 1 adopt the parametrization allowing the simplest equations, in the implementation and comparison all parameters have been normalized within the interval  $[0, 1]$ .



**Figure 1** - Two of the four types of curves considered in the comparative evaluation: (a) curve generated by splines and (b) curve synthesized in terms of Fourier components of curvature.

A B-spline curve is generated by combining a series of blending functions  $B_{j,r}$  weighted by a set of control points  $p_j$  [Mortenson, 1985]. Particularly, cubic B-splines may be expressed in piecewise fashion as:

$$p_j(t) = \frac{1}{6} \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_{j-1} \\ p_j \\ p_{j+1} \\ p_{j+2} \end{bmatrix}$$

where  $p_j(t) = (x_j(t), y_j(t))$ ,  $0 \leq t < 1$ ,  $i = 1, 2, \dots, n-2$ . From this B-spline representation it is possible to obtain the analytical derivatives of the curve, i.e.:

$$\dot{p}_i(u) = (\dot{x}_i(u), \dot{y}_i(u))$$

$$\ddot{p}_i(u) = (\ddot{x}_i(u), \ddot{y}_i(u))$$

The next important step in devising a fair and meaningful comparative framework regards the means through which the adopted curves are spatially sampled onto the orthogonal lattice. Such a choice should reflect the inherent features of each specific situation — for instance, a CCD camera can be roughly modeled by using the square-box quantization scheme, which is the scheme adopted in the present article. Such a spatial quantization, however, implies lost of information through a degenerated mapping. Indeed, there is an infinity of possible continuous curves having the same discretizations, hence the necessity of Gaussian smoothing as a regularizing constraint. Indeed, such low-pass act in such a way as to limit the infinite bandwidth implied by the point singularities introduced in the square-box quantization. Although providing a model of contours typically found in images, the use of spatial quantization also implies the problem of how to compare the estimated curvature with the analytic exact

curvature of the pre-image (i.e. the original curve before quantization). The chosen criterion is to compare the curvature at each lattice point with the closest analytic curvature obtained by a fine but discrete quantization in the original parameter.

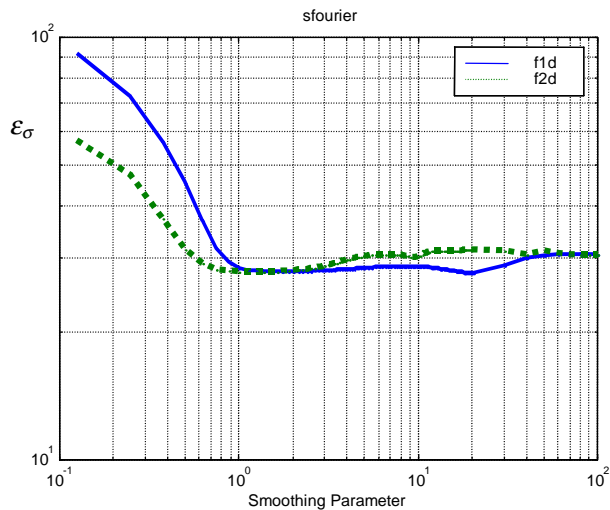
The next important consideration relates to how the scale parameters in both curvature estimation methods should be taken into account. While ideally it would be expected that each method produced accurate curvature estimations irrespectively to the parameter values, this is by no means feasible in practice. Indeed, the parameters in each method arise from the need to implement regularization of the spatially sampled curves, and are thus unavoidable. If on one hand large values of  $\sigma$  will produce more smoothed representations by filtering out the quantization noise, too high values for this parameter will also remove inherent information from the original curves. It is thus clear that the best estimations should be obtained within an interval of spatial scale where the respective smoothing is just enough to filter out the noise and not yet too strong as to undermine the curve representation. For such reasons, we aimed at investigating how the curvature estimation error varies with the scale parameter  $\sigma$ .

## 5 RESULTS

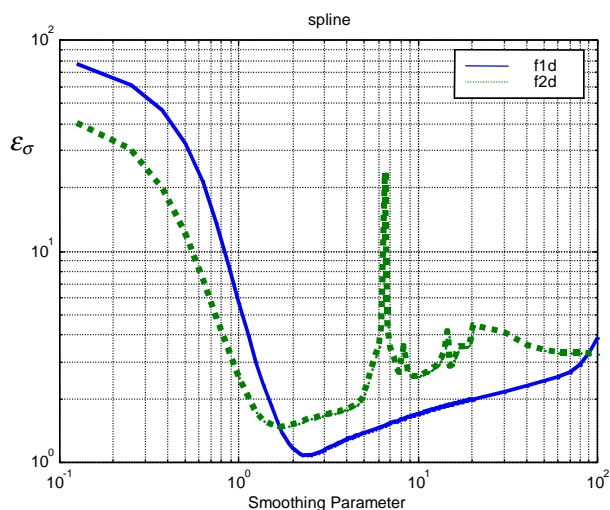
The evaluation framework was developed in MATLAB and Borland Delphi, and included database and visualization facilities. The software was executed on a Pentium 233. The curvature of each considered curve was estimated for  $\sigma$  varying from 0 to 100, and comprehensive data statistics were obtained including: (a) execution times; (b) evolution of square error  $\epsilon_\sigma$  between analytic and estimated curvature values; and (c) histograms of curvature estimation error in terms of the scale parameter and curvature intervals. Average execution times of 2s and 120s were obtained for the 1-D and 2-D approaches, respectively. Regarding the curvature estimation error (square

distances between analytical and estimated values), it was observed that large errors are found for very small  $\sigma$ , and that such errors tend to reach their minimum values around  $\sigma = 2.5$ . As the scale parameter increases, the error start to grow, reaching a stable value for  $\sigma$  larger than 200. While this behavior was observed for virtually every considered curve, the fined details of the error evolution do vary from case to case, depending of the specific curves. Figure 2 shows the error evolution for the curves in Figure 1(a) and (b).

Another interesting phenomenon was the fact that the 2-D estimation sometimes produced some instabilities, such as in Figure 2(a), characterized by sudden increase of the error value.



(a)



(b)

Figure 2 - The square error between the analytic and estimated curvatures in terms of the scale parameter  $\sigma$ .

## 6 DISCUSSION

In addition to corroborating the validity of Fourier-based approaches to multi-resolution curvature estimation of closed parametric contours, the obtained results have also indicated

some interesting phenomena and tendencies. To start with, every considered situation was characterized by presenting minimum error at about  $\sigma = 2.5$ . The reason for this is that for a smaller smoothing degree, the noise implied by the spatial quantization of the original curves still prevails; for larger values of the scale parameter, the smoothing is too high and the curve becomes too much distorted. Fortunately, small curvature estimation errors are obtained for a reasonably large interval around  $\sigma = 2.5$ .

The interference effect in the 2-D approach has been interpreted as being understood as being caused by two portions of the curve with distinct curvature values that are close in the 2-D space but distant in the 1-D parameter domain, such as the bottleneck in Figure 1(a). Since the derivatives in Equation (6) are implemented in the 2-D space, such interferences tend to mix the curvatures at both portions of the contour. The error peak has been verified to occur at a spatial scale proportional to the minimal 2-D distance between portions of the same curve. Such an effect indicates a definite advantage of the 1-D scheme, since such interferences are completely avoided. Allied with the much faster execution speed, such behavior provides a clear indication of the superiority of the 1-D method. However, it should be observed that the 2-D method is not much worse and represents a valid alternative in situations where the curvature in every point of the 2-D domain have to be estimated.

## 7 CONCLUSIONS AND FUTURE DEVELOPMENTS

This article has addressed the important problem of accurate numerical estimation of point curvature in closed parametrized curves. While the application potential of such techniques is considerable for vision research, image processing and analysis, shape analysis and computer graphics, relatively little attention has been focused on the issue of comparing and validating such numerical approaches. The present article has not only introduced a new approach to curvature estimation, namely the 2-D Fourier-based methodology, as well as the exclusion of perimeter/energy normalization, but also presented a careful and reasonably comprehensive comparison of the 1-D and 2-D Fourier-based approaches. An interesting phenomenon that has been verified consists in the fact that the 2-D approach tend to suffer when the curve is characterized by regions that are close in the 2-D space but distant along the 1-D parameter. Although the obtained results clearly indicate definite advantages in using the 1-D as far as execution time and accuracy is concerned, the 2-D approach is still relevant for problems involving the estimation of curvature in 2-D embeddings of parametric spatial curves. One such problem, namely the use of level-set methods to calculate propagating fronts [Sethian, 1996], is currently under investigation in the Cybernetic Vision Research Group, including the possibility of using the 1-D and 2-D curvature estimation techniques addressed in the current article as means for improving the accuracy of the curvature values required by the level-set approaches.

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